Rapid Filters and Guided Gregorieff Forcing

David Chodounský
Charles University in Prague

January 31, 2010

Definition

A filter $\mathcal{F}$ is a *p-filter* if for each $D \in [\mathcal{F}]^\omega$ there exists $p \in \mathcal{F}$ such that $p \subseteq^* d$ for each $d \in D$. The set p is called *pseudointersection* of D. An ideal $\mathcal{I}$, for which the dual filter $\mathcal{I}^*$ is an p-filter, is *p-ideal*.

Definition

A filter $\mathcal{F}$ is a *p-filter* if for each $D \in [\mathcal{F}]^\omega$ there exists $p \in \mathcal{F}$ such that $p \subseteq^* d$ for each $d \in D$. The set p is called *pseudointersection* of D. An ideal $\mathcal{I}$, for which the dual filter $\mathcal{I}^*$ is an p-filter, is *p-ideal*.

Definition

Let $\mathcal{I}$ be an ideal on $P(A)$. $\mathcal{I}$ is an *tall* ideal if for each $b \in [P(A)]^\omega$ there exist some $a \in \mathcal{I} \cap [b]^\omega$.

Lemma (M. Talagrand)

For a filter $\mathcal{F}$ in $P(\omega)$ the following are equivalent:

1. *$\mathcal{F}$ is non-meager subset of $P(\omega)$.*
2. *$\mathcal{F}$ is unbounded (i.e. enumerating functions of sets in $\mathcal{F}$ are unbounded subset of $({}^{\omega}\omega, <^*)$.)*
3. *For each decomposition of $\omega = \bigcup I_n$ into intervals there is a set $F \in \mathcal{F}$ such that $F \cap I_n = \emptyset$ for infinitely many intervals.*

Lemma (M. Talagrand)

For a filter $\mathcal{F}$ in $P(\omega)$ the following are equivalent:

1. *$\mathcal{F}$ is non-meager subset of $P(\omega)$.*
2. *$\mathcal{F}$ is unbounded (i.e. enumerating functions of sets in $\mathcal{F}$ are unbounded subset of $({}^{\omega}\omega, <^*)$.)*
3. *For each decomposition of $\omega = \bigcup I_n$ into intervals there is a set $F \in \mathcal{F}$ such that $F \cap I_n = \emptyset$ for infinitely many intervals.*

Lemma (A. Miller ?)

For a filter $\mathcal{F}$ in $P(\omega)$ the following are equivalent:

1. *$\mathcal{F}$ is rapid.*
2. *Enumerating functions of sets in $\mathcal{F}$ are dominating family.*
3. *There exist an increasing function $f \in {}^{\omega}\omega$ such that for every sequence $\{t_i : t_i \in [\omega]^{<\omega}, i \in \omega\}$ there exists some $F \in \mathcal{F}$ such that $|F \cap t_i| < f(i)$ for each $i \in \omega$.*

Definition (Gregorieff's forcing)

Let $\mathcal{F}$ be a filter on ω. Put

$$G(\mathcal{F}) = (\{g : I \to 2; I \in \mathcal{F}^*\}, \supset) .$$

The forcing notion $G(\mathcal{F})$ is called *Gregorieff's forcing*.

Definition (Gregorieff's forcing)

Let $\mathcal{F}$ be a filter on ω. Put

$$G(\mathcal{F}) = (\{g : I \to 2; I \in \mathcal{F}^*\}, \supset)\,.$$

The forcing notion $G(\mathcal{F})$ is called *Gregorieff's forcing*.

Definition (Guided Gregorieff's forcing)

Let $\mathcal{T} = \{T_\alpha : T_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ be a strictly increasing tower, i.e. $T_\alpha \subset^* T_\beta$ and $|T_\beta \setminus T_\alpha| = \omega$ for $\alpha < \beta \in \omega_1$.
Denote $\mathcal{A} = \{A_\alpha = T_{\alpha+1} \setminus T_\alpha, \alpha \in \omega_1\}$. For $F = \{f_\alpha : A_\alpha \to 2\}$ the forcing notion $P(\mathcal{T}, F)$ consists of pairs (g, β) where $\beta \in \omega_1$ and g is a function with domain $D(g) =_* T_\beta$ to 2. Moreover $(g, \beta) \in P(\mathcal{T}, F)$ iff $g \restriction A_\alpha =_* f_\alpha$ for each $\alpha < \beta$.
The ordering is reversed inclusion, $(g, \beta) \leq (h, \gamma)$ iff $h \subseteq g$.

Lemma

1. *Let $\mathcal{F}$ be a non-meager p-filter on ω. Then the Gregorieff's forcing G($\mathcal{F}$) is proper and ${}^{\omega}\omega$ bounding.*
2. *Let $\mathcal{T}$ be a strictly increasing tower of size ω_1 which generates a non-meager ideal. Then $P(\mathcal{T}, F)$ is a proper ${}^{\omega}\omega$ bounding forcing notion for each choice of F.*

Lemma

1. *Let $\mathcal{F}$ be a non-meager p-filter on ω. Then the Gregorieff's forcing G($\mathcal{F}$) is proper and ${}^{\omega}\omega$ bounding.*
2. *Let $\mathcal{T}$ be a strictly increasing tower of size ω_1 which generates a non-meager ideal. Then $P(\mathcal{T}, F)$ is a proper ${}^{\omega}\omega$ bounding forcing notion for each choice of F.*

Lemma

1. *Let $\mathcal{F}$ be a rapid p-filter on ω. Then the Gregorieff's forcing G($\mathcal{F}$) has Sacks property.*
2. *Let $\mathcal{T}$ be a strictly increasing tower of size ω_1 such that the filter $\mathcal{F}$ dual to the ideal generateted by $\mathcal{T}$ is rapid. Then the forcing $P(\mathcal{T}, F)$ has Sacks property for each choice of F.*

Definition (p-filter game)

Let $\mathcal{F}$ be a filter in $P(\omega)$ containing no finite sets. The following game is called *p-filter game* $G_{\mathcal{F}}$. In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with its finite subset $s_n \in [F_n]^{<\omega}$. After ω many moves player II wins if $\bigcup\{s_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

Definition (p-filter game)

Let $\mathcal{F}$ be a filter in $P(\omega)$ containing no finite sets. The following game is called *p-filter game* $G_{\mathcal{F}}$. In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with its finite subset $s_n \in [F_n]^{<\omega}$. After ω many moves player II wins if $\bigcup\{s_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

Lemma (Laflamme)

$\mathcal{F}$ is a non-meager p-filter in $P(\omega)$ if and only if player I has no winning strategy in game $G_{\mathcal{F}}$.

Lemma

Let $\mathcal{F}$ be a rapid p-filter. There exists an increasing function $f \in {}^{\omega}\omega$ such that for each strategy of player I for the p-filter game $G_{\mathcal{F}}$ there exists some sequence $\{s_i : |s_i| < f(i), i \in \omega\}$ of moves of player II which beats this strategy.

Definition (strong-Q-sequence, J. Steprans)

Let $\mathcal{A}$ be a subset of a Boolean algebra A. $\mathcal{A}$ is a *strong-Q-sequence* in A if the following holds true: For every $F : \mathcal{A} \to A$ such that $F(a) \leq a$ there exists $c \in A$ such that $c \wedge a = F(a)$ for each $a \in \mathcal{A}$.

Fact

Every strong-Q-sequence in A is an antichain and if A is κ-complete than the converse is also true for antichains of size less than κ.

$$\mathcal{A}^{\perp} = \{x \in A : x \wedge a = 0 \text{ for each } a \in \mathcal{A}\}.$$

Lemma (J. Steprans)

Let A and B be Boolean algebras. Suppose that $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ are strong-Q-sequences. Suppose furthermore that there is a bijection $\Psi : \mathcal{A} \to \mathcal{B}$ such that $A \restriction a$ is isomorphic to $B \restriction \Psi(a)$ for each $a \in \mathcal{A}$. Then $A/\mathcal{A}^{\perp}$ is isomorphic to $B/\mathcal{B}^{\perp}$.

Proof.

For each $a \in \mathcal{A}$ fix an isomorphism $\psi_a : A \restriction a \to B \restriction \Psi(a)$. For $x \in B$ and $a \in \mathcal{A}$ put $G_x(a) = \psi_a^{-1}(x \wedge \Psi(a))$. Define isomorphism $\theta : B/\mathcal{B}^{\perp} \to A/\mathcal{A}^{\perp}$ by the rule $\theta(x) \wedge [a] = [G_x(a)]$ for each $a \in \mathcal{A}$.

Definition (strong-Q-sequence of size ω_1 in $P(\omega)/fin$)

$\mathcal{A} = \{A_\alpha : A_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ is a *strong-Q-sequence* (of size ω_1) iff for each $F = \{f_\alpha : A_\alpha \to 2\}$ there exists $f_F : \omega \to 2$ such that $f_F \restriction A_\alpha =_* f_\alpha$.

Definition (strong-Q-sequence of size ω_1 in $P(\omega)/fin$)

$\mathcal{A} = \{A_\alpha : A_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ is a *strong-Q-sequence* (of size ω_1) iff for each $F = \{f_\alpha : A_\alpha \to 2\}$ there exists $f_F : \omega \to 2$ such that $f_F \restriction A_\alpha =_* f_\alpha$.

Definition (Q-set of size ω_1 in $P(\omega)$)

$\mathcal{A} = \{A_\alpha : A_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ is a *Q-set* (of size ω_1) iff for each $F = \{f_\alpha : A_\alpha \to 2, f_\alpha \text{ constant}\}$ there exists $f_F : \omega \to 2$ such that $f_F \restriction A_\alpha =_* f_\alpha$.

Corollary

If a strong-Q-sequence of size ω_1 in $P(\omega)/fin$ exists, then $2^\omega = 2^{\omega_1}$.

Example

Pick B_α, $\alpha \in \omega_1$ distinct maximal branches throught ${}^{<\omega}2$. These B_α's (as sets of nodes) form an AD system which is not a strong-Q-sequence. (Put $f_\alpha(s) = i$ iff $s^\frown\{i\} \in B_\alpha$.)
Note that $\mathrm{MA}_{\omega_1}(\sigma - centered)$ implies that $\{B_\alpha, \alpha \in \omega_1\}$ is a Q-set.

Example

Pick B_α, $\alpha \in \omega_1$ distinct maximal branches throught ${}^{<\omega}2$. These B_α's (as sets of nodes) form an AD system which is not a strong-Q-sequence. (Put $f_\alpha(s) = i$ iff $s^\frown\{i\} \in B_\alpha$.)
Note that $\mathrm{MA}_{\omega_1}(\sigma-\mathit{centered})$ implies that $\{B_\alpha,\, \alpha \in \omega_1\}$ is a Q-set.

Theorem (J. Steprans)

$MA_{\omega_1}(\sigma-\mathit{linked})$ implies that there is no strong-Q-sequence of size ω_1 in $P(\omega)/\mathit{fin}$.

Theorem (J. Steprans)

It is consistent that there exist a strong-Q-sequence of size ω_1 in $P(\omega)/\mathit{fin}$.

Katowice problem

Question (Katowice problem)

Is it consistent with ZFC that $P(\omega)/fin \simeq P(\omega_1)/fin$?

Katowice problem

Question (Katowice problem)

Is it consistent with ZFC that $P(\omega)/fin \simeq P(\omega_1)/fin$?

Theorem (B. Balcar)

If $P(\omega)/fin \simeq P(\omega_1)/fin$ then $\mathfrak{d} = \omega_1$ (and $2^\omega = 2^{\omega_1}$).

Proof.

Fix a homeomorphism $h : P(\omega_1)/fin \to P(\omega)/fin$.
$\omega_1 = \bigcup\{B_n, n \in \omega\}$, where $|B_n| = \omega_1, B_n \cap B_m = \emptyset$ for $n \neq m \in \omega$
Fix $\{A_n, n \in \omega\}$ such that $\omega = \bigcup\{A_n, n \in \omega\}, [A_n] = h([B_n])$ and $A_n \cap A_m = \emptyset$ for $n \neq m \in \omega$.
Let $b : \omega \to \omega \times \omega$ be a bijection, $b''(A_n) = \{n\} \times \omega$.
For $\alpha < \omega_1$ fix D_α such that $[D_\alpha] = h([\omega_1 \setminus \alpha])$.
Put $f_\alpha(n) = \min\{i \in \omega : (n, i) \in b''(D_\alpha)\}$.
$\{f_\alpha : \alpha \in \omega_1\}$ is a ω_1-scale.

What are properties of the ideal $[\omega]^\omega$ in $P(\omega_1)/fin$?

What are properties of the ideal $[\omega]^\omega$ in $P(\omega_1)/fin$?

It is a tall ideal.

What are properties of the ideal $[\omega]^\omega$ in $P(\omega_1)/\mathit{fin}$?

It is a tall ideal.

It is generated by an increasing ω_1 sequence

$$\{t_\alpha = [\alpha \cdot \omega] : \alpha \in \omega_1\},$$

and $\{t_{\alpha+1} - t_\alpha : \alpha \in \omega_1\}$ is a strong-Q-sequence.

What are properties of the ideal $[\omega]^\omega$ in $P(\omega_1)/fin$?

It is a tall ideal.

It is generated by an increasing ω_1 sequence

$$\{t_\alpha = [\alpha \cdot \omega] : \alpha \in \omega_1\},$$

and $\{t_{\alpha+1} - t_\alpha : \alpha \in \omega_1\}$ is a strong-*Q*-sequence.

It has has nonempty intersection with all *p*-ultrafilters.

Fact

Every p-ultrafilter $\mathcal{F}$ on ω_1 contains a countable set.

Proof.

$\mathcal{F}$ in not σ-complete so fix $D \in [\mathcal{F}]^\omega$ such that $\bigcap D = \emptyset$. There exists $p \in \mathcal{F}$ such that $p \subseteq^* d$ for each $d \in D$. This implies that $|p| = \omega$.

What are properties of the ideal $[\omega]^\omega$ in $P(\omega_1)/fin$?

It is a tall ideal.

It is generated by an increasing ω_1 sequence

$$\{t_\alpha = [\alpha \cdot \omega] : \alpha \in \omega_1\},$$

and $\{t_{\alpha+1} - t_\alpha : \alpha \in \omega_1\}$ is a strong-*Q*-sequence.

It has has nonempty intersection with all *p*-ultrafilters.

Definition

An increasing tower $\mathcal{T}$ in $P(\omega)$ is a *C-tower* iff the following holds.

1. $\mathcal{T} = \{T_\alpha : \alpha \in \omega_1\}$
2. $\mathcal{T}$ generates a tall ideal.
3. $\{T_{\alpha+1} \setminus T_\alpha : \alpha \in \omega_1\}$ is a strong-Q-sequence.
4. $\mathcal{T}$ has nonempty intersection with each nonprincipal p-ultrafilter.

Fact

If $P(\omega)/fin \simeq P(\omega_1)/fin$ then there is a C-tower.

Theorem

It is consistent that there exist a C-tower and $\mathfrak{d} = \omega_1$.

Start in a model of ZFC + $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$. Pick $\mathcal{T} = \{T_\alpha : T_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ a strictly increasing tower; $T_\alpha \subset^* T_\beta$ and $|T_\beta \setminus T_\alpha| = \omega$ for $\alpha < \beta \in \omega_1$. Suppose moreover that $\mathcal{T}$ generates a non-meager ideal and denote $\mathcal{F}$ the dual non-meager *p*-filter, $\mathcal{A} = \{A_\alpha = T_{\alpha+1} \setminus T_\alpha, \alpha \in \omega_1\}$.
The desired model will be obtained by countable support iteration of length ω_2 of proper ${}^\omega\omega$ bounding forcing notions. Start with $G(\mathcal{F} \times \emptyset)$ and in step $\beta \in \omega_2$ force with $P(\mathcal{T}, F^\beta)$ for suitably chosen $F^\beta = \{f_\alpha : A_\alpha \to 2, \alpha \in \omega_1\}$.

Note that in the resulting model

$$P(\omega_1)/\{t_{\alpha+1} - t_\alpha : \alpha \in \omega_1\}^\perp \simeq P(\omega)/\mathcal{A}^\perp.$$

where $\{t_\alpha = [\alpha \cdot \omega]$ for $\alpha \in \omega_1\}$

Theorem (H. Judah, S. Shelah)

It is consistent that there exists a Q-set of size ω_1 *and a set of reals of size* ω_1 *which has not measure zero.*

Theorem (H. Judah, S. Shelah)

It is consistent that there exists a Q-set of size ω_1 *and a set of reals of size* ω_1 *which has not measure zero.*

To prove this result we need a forcing which introduces a Q-set and has Sacks property. This can be easily done by iterating Guided Gregorieff forcing notions $P(\mathcal{T}, F^{\beta})$ where $\mathcal{T}$ generates a rapid ideal (we can even get a strong-Q-sequence).

- J. Steprans, *Strong Q-sequences and variations on Martins Axiom*, Canad. J. Math. **37** (1985) no. 4, 730–746.
- W. W. Comfort, *Compactifications: recent results from several countries*, Topology Proc. **2** (1977), no. 1, 61–87 (1978).
- H. Judah, S. Shelah, *Q-sets, Sierpiński sets, and rapid filters*, Proc. Amer. Math. Soc. **111** (1991) 821–832.
- P. Nyikos, *Cech-Stone remainders of discrete spaces*, in: Open Problems in Topology II, Elliott Pearl, ed., Elsevier B.V., Amsterdam, 2007, 207–216.